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Results and problems on Dedekind sums

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Abstract

Under the influence of B. Riemann, R. Dedekind was interested in the behavior of the function $\eta(z)$, defined by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right), \ \mathrm{Im}(z) > 0,$$

It should be noted that previously, Jacobi and Hermite had already considered this function in their work. However, it is R. Dedekind who studied it the most. More specifically, he examined the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the Poincaré half-plane. Precisely, under the action of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$. He discovered an important formula involving the sums

$$s(d, c) = \sum_{k=1}^{|c|} \left(\left(\frac{k}{c}\right)\right) \left(\left(\frac{kd}{c}\right)\right)$$

commonly referred to as Dedekind sums, where $((x))$ denotes the "sawtooth" function. These sums play a crucial role in number theory. In this presentation, we will explore both historical results and recent advances regarding this topic. We will highlight the fundamental arithmetic properties of these sums, notably Dedekind's reciprocity law, as well as notions of density and equidistribution modulo 1. We will also establish connections between these sums and the special values of the partial zeta function associated with a number field, as well as the Euler class in cohomology. If time permits, I will also address three analogues of the Dedekind sums. These analogues represent multidimensional and elliptic generalizations that broaden our understanding of the properties and applications of Dedekind sums in more complex contexts. Furthermore, I will present some open problems that remain fascinating in number theory.

Keywords: *Dedekind sums, Number theory, Modular group, Zeta function, Reciprocity law, Equidistribution.*

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